

**Economics Division  
University of Southampton  
Southampton SO17 1BJ, UK**

**Discussion Papers in  
Economics and Econometrics**

**Title: The Role of Curvature in the  
Transformation Frontier between Consumption  
and Investment**

By : Alessandro Mennuni (University of Southampton),

No. 1407

**This paper is available on our website  
<http://www.southampton.ac.uk/socsci/economics/research/papers>**

# The Role of Curvature in the Transformation Frontier between Consumption and Investment<sup>\*†</sup>

Alessandro Mennuni  
University of Southampton

March 23, 2014

## Abstract

The vast majority of the business cycle literature assumes a linear transformation frontier between consumption and investment goods. This assumption neglects a relationship, present in the data, between the relative demand of consumption and investment, and its relative price. This assumption also leads to counterfactual saving rates. A simple extension of the real business cycle model is proposed where the transformation frontier can be concave. Alternative identification strategies lead to the estimation of a concave frontier, with a dramatic improvement of the prediction of the saving rate.

**JEL Classification Codes:** E13, E21, E32

**Keywords:** Business Cycle; Investment-Specific Shocks; Sectorial Reallocation

---

\*I thank my advisor Martin Gervais, John Knowles, Paul Klein, Xavier Mateos-Planas, Roman Sustek, Juan Carlos Conesa, Salvador Ortigueira, Ramon Marimon, Sarolta Laczó, Yang Lu, Leonardo Melosi, Maxym Kryshko, Marco Bassetto, Mariacristina De Nardi and workshop participants at UCL, the University of Southampton and the Max Weber conference at the EUI for helpful comments. All remaining errors are mine.

†An earlier version of the paper was entitled “The Role of Curvature in the Transformation Frontier for Measuring Technology Shocks”

# 1 Introduction

Recent work in the business cycle literature is based on the neoclassical growth framework (Kydland and Prescott (1982)). Greenwood et al. (1988) showed that shocks to the productivity of investment goods (I-shock) are an important source of fluctuations, together with neutral or total factor productivity shocks that hit all sectors of the economy (N-shock). This recognition engendered several studies of this mechanism, including Greenwood et al. (2000), Cummins and Violante (2002), Fisher (2006) and Smets and Wouters (2007). I-shocks are now embedded in the vast Dynamic Stochastic General Equilibrium (DSGE) literature.

To identify the I-shock, these papers often use the fact that, under the assumption of a linear transformation between consumption and investment, the relative price of investment goods only moves with I-shocks. This is a key identification assumption because, without identifying investment shocks from the price equation, Justiniano et al. (2010) find that the I-shock should be 4 times more volatile in order to match business cycle fluctuations. This sharp contrast calls for an investigation of this price equation, which, despite its wide use, remains largely under-investigated. While there are countless ways in which this price equation can be modified—see Floetotto et al. (2009) and Justiniano et al. (2011) for a discussion—this paper identifies two signs of misspecification which are used to discipline this task:

(i) The two shocks identified through the above framework are strongly negatively correlated.<sup>1</sup>

(ii) When simulated with the identified shocks, the model's prediction of the saving rate is grossly counterfactual.<sup>2</sup>

The paper argues that these two observations are related and indicate a concave frontier.

---

<sup>1</sup>After removing unit roots from the shocks, I find a significant correlation between the two shocks of -22%. This result is consistent with Schmitt-Grohe and Uribe (2011). However, they take a different approach: rather than as a sign of misspecification, they interpret the relationship between the shocks as a genuine property and embed it in the exogenous productivity processes.

<sup>2</sup>In the model of this paper, the saving rate is equivalent to the investment-output ratio.

Fact (i)—that the two shocks identified when assuming a linear frontier are negatively correlated—is a sign of concavity in the transformation frontier for the following reason: a concave frontier implies a positive relation between the relative price and the N-shock.<sup>3</sup> If this relationship is present but neglected, one would have to wrongly attribute the increase in the relative price that comes after a positive N-shock to a decrease in the I-shock: every time the price increases as a consequence of a positive N-shock, a researcher armed with the simplified price equation would impute the increase in the price to a decrease in the I-shock.<sup>4</sup> This would make the two shocks appear negatively correlated.

To assess the model’s prediction of the saving rate (fact ii), the model is simulated with the shocks and initial conditions identified from the data. This procedure is not common in the literature, where the identified shocks are only used to estimate their stochastic processes, and then simulations consist in drawing from these processes, often ignoring the correlation between the innovations, and simulating around the balanced growth path. Instead, using the actual realizations of the shocks allows the model to be tested by direct comparison of the time series as is standard in regression analysis. Notably, the  $R^2$  of the true and predicted saving rates is negative. This suggests that the simple mean is a better predictor of the saving rate time series. It is important to notice that this result is closely related to the negative correlation between the shocks (fact i). This is because the spurious negative investment shocks associated with positive N-shocks induces a counterfactual decrease in the saving rate predicted by the model.

While the literature suggested ways to increase comovement, ranging from altering preferences ([Greenwood et al. \(1988\)](#)) have to rely on very low short-run wealth effects in the labor supply, as recently emphasized by [Jaimovich and Rebelo \(2009\)](#)) to abandoning the one sector model all together ([Guerrieri](#)

---

<sup>3</sup>This is due to the fact that a positive N-shock increases the saving rate (the share of output not consumed) because of the positive effect on the return on capital; firms, due to the concave transformation frontier, are induced to meet the demand shift to investment goods through an increase in their relative price.

<sup>4</sup>With a linear frontier, at any interior solution firms are indifferent between the amount of consumption and investment goods to produce. Therefore, the demand shift would not induce a price shift.

et al. (2009)), this paper shows how the key ingredient is a concave frontier, a feature which has been largely abstracted from in the literature.<sup>5</sup> An exception is Fisher (1997), who notices counterfactual negative comovement of household investment (durable goods and housing) with other expenditures after a N-shock. He proposed a concave frontier between household and business investment to address the issue. Perhaps because the DSGE literature maintained a higher level of aggregation, that paper was not as influential as it could have been to point to the importance of curvature. Furthermore, I-shocks were not commonly embedded in the literature at the time: with only a N-shock, consumption and investment have the right comovement as shown, for example by Plosser (1989) (albeit a counterfactual constant price). While it remains valuable to work at such a level of disaggregation, the comovement problem—now evident at a more aggregate level—prompts an investigation of concavity between non durable consumption and total investment; the usual distinction adopted in the DSGE literature. A further motivation to measuring concavity in the frontier between these two aggregates comes from the literature on news shocks: Beaudry and Portier (2007) show that a neoclassical economy with a concave frontier between consumption and investment can generate news driven business cycles with comovement.

To highlight the role of a concave frontier, the model is enhanced with curvature in the transformation frontier by adding only one parameter to the original framework. This is convenient in that it allows the one-sector characterization of the original framework to be preserved, and the same data to be used to fully parameterize the model, for a transparent comparison.

Since it is argued that the lack of curvature is the reason for the above-mentioned drawbacks, one possible way to estimate the curvature is to choose it so that the two shocks appear uncorrelated, in order to avoid capturing as an I-shock the increase in the price due to the N-shock. Alternatively, one can exploit fact (ii) and pick the curvature to maximize the fit of the saving rate.

---

<sup>5</sup>Even large-scale DSGE models with the typical “bells and whistles” considered in the literature struggle with the problem of generating the right comovement between consumption and investment. For example, Justiniano and Primiceri (2008) show that the comovement between consumption and investment is positive in the data and negative in the model.

Strikingly, the curvature under the two strategies is very close and leads to the same implications. In particular, with both strategies the model improves dramatically in its prediction of the saving rate, while matching the long run great ratios and second moment conditions as in the linear framework.

The finding that the production possibility frontier is concave contrasts with [Schmitt-Grohe and Uribe \(2011\)](#), who find no evidence of a concave frontier. However, they allow their model to have a concave frontier by assuming that the investment technology is concave while that of consumption is linear. In this way, concavity affects long-run trends, for which the linear framework is very successful. This tension between long run trends and a concave frontier may be what is leading their likelihood criterium to reject concavity all together. Their contribution motivates the strategy developed in this paper: to model curvature in a way that has short run implications, while preserving the growth properties of the linear framework, which is able to reconcile a positive investment-specific trend and decreasing relative prices with a balanced growth path as shown by [Greenwood et al. \(1997\)](#).

To gain insight into the forces driving the identification of the curvature and its implications, the model is kept very simple; attention is restricted to neutral and investment shocks, and the frictions usually included by the recent literature are not considered in the first sections. Then the paper compares the effects of a concave frontier with those that come from adding capital adjustment costs, which intuitively could have similar implications. This friction only induces a negligible improvement in the saving rate prediction of the original model with a linear frontier and does not affect the identification of the I-shock.<sup>6</sup> Results are also robust to the introduction of habit persistence and capital utilization. To clearly show the identification of the curvature, the empirical sections used calibration and GMM techniques. The final section, however, estimates the model with all the frictions with Bayesian methods, as

---

<sup>6</sup>This friction introduces *inter-temporal* adjustment costs. Instead, concavity in the transformation frontier is a concept that is closer to the *intra-temporal* adjustment costs between capital goods considered by [Huffman and Wynne \(1999\)](#) and [Valles \(1997\)](#). [Kim \(2003\)](#) show equivalence results between the two frictions with fix labor supply. These results do not pertain to the identification of I-shocks. In fact, the paper shows that capital adjustment costs have no implications for the relative price and the identification of I-shocks.

is now standard in the literature. Even assuming a uniform prior distribution on the curvature parameter, the frontier is estimated to be strictly concave.

The paper is organized as follows: the next section identifies and discusses the misspecification, Section 3 modifies the framework in order to allow for curvature in the transformation frontier, Section 4 illustrates the findings, Section 5 extends the model to other frictions, Section 5.1 shows the bayesian estimation and Section 6 concludes. The Appendix contains data sources, the equilibrium conditions and equivalent de-trended specifications.

## 2 Identifying the Misspecification

Below follows a description of the standard real business cycle model with investment-specific technological change like, for instance, the one adopted in Fisher (2006). The set-up is written in a way that highlights the role played by the linear frontier and lends itself to a natural extension with a concave frontier. The representative household solves the following problem, taking prices as given:

$$\begin{aligned} \max_{\{c_t, k_{t+1}, n_t\}} E_0 \left[ \sum_{t=0}^{\infty} \beta^t \left( \log(c_t) - \xi \frac{n_t^{1+1/\nu}}{1+1/\nu} \right) \right] \\ s.t. \quad c_t + p_t k_{t+1} = w_t n_t + p_t k_t (1 + r_t - \delta). \end{aligned}$$

$E_0$  is the (rational) expectation operator about prices given information at  $t = 0$ . These preferences are adopted for instance by Ríos-Rull et al. (2009).  $\nu$  is the Frisch elasticity of hours,  $n_t$ .  $\xi$  scales the cost of working and it determines the average level of hours.  $\beta$  is the discount factor.  $c_t$  is consumption,  $p_t$  the price of capital in terms of consumption goods.  $w_t$  is the wage rate and  $r_t$  the rental price of capital,  $k_t$ .  $\delta$  is the rate at which capital depreciates. Production takes place through a constant returns to scale Cobb-Douglas technology and capital evolves according to the law of motion

$$k_{t+1} - k_t(1 - \delta) = V_t A_t k_t^\alpha n_t^{1-\alpha} s_t,$$

while non-durable consumption is

$$c_t = (1 - s_t)A_t k_t^\alpha n_t^{1-\alpha}, \quad (1)$$

where  $s_t$  is the fraction of physical production allocated to investment.  $V_t$  is the investment shock, which only hits the production devoted to increasing the capital stock.  $A_t$  is a neutral shock that hits both sectors in the same way. Firms are competitive and can choose whether to sell production as consumption or capital: given prices, they solve the following static problem:

$$\begin{aligned} \max_{k_t, n_t, s_t \in [0,1]} & y_t - w_t n_t - p_t k_t r_t \\ \text{s.t.} & \end{aligned}$$

$$y_t = (1 - s_t)A_t k_t^\alpha n_t^{1-\alpha} + p_t V_t s_t A_t k_t^\alpha n_t^{1-\alpha}. \quad (2)$$

The first order conditions for the firm are as below:

$$\begin{aligned} \alpha A_t k_t^{\alpha-1} n_t^{1-\alpha} (1 - s_t + p_t V_t s_t) &= p_t r_t \\ (1 - \alpha) A_t k_t^\alpha n_t^{-\alpha} (1 - s_t + p_t V_t s_t) &= w_t \end{aligned}$$

and for an interior  $s_t$

$$p_t = 1/V_t. \quad (3)$$

The price equation (3) reflects the fact that a firm can choose where to allocate its inputs with no costs. Hence, it will be indifferent between producing consumption or investment goods if and only if (3) holds. This implication of the model is what is disputed in the present paper.

From (1), (2) and (3),  $s_t = 1 - \frac{c_t}{y_t}$  holds. Therefore total output simplifies to

$$y_t = A_t k_t^\alpha n_t^{1-\alpha}.$$

From this and (3), time series for  $A$  and  $V$  are identified as follows:

$$A_t = \frac{y_t}{k_t^\alpha n_t^{1-\alpha}}, \quad (4)$$

$$V_t = 1/p_t. \quad (5)$$

## 2.1 Correlation Between Shocks

The data are constructed by extending to 2012 II the data-set in [Ríos-Rull et al. \(2009\)](#). In particular, data on the relative price of investment goods extend those constructed by [Gordon \(1990\)](#), and successively by [Cummins and Violante \(2002\)](#) and [Fisher \(2006\)](#). The dataset starts in 1948 I and the sources are detailed in Appendix [A.1](#). The two shocks are identified through equations (4)–(5). To identify the neutral shock,  $\alpha$  is assumed to be equal to 0.36, which is consistent with the empirical labor share. The results in this section are robust to changes in this parameter.

ADF and Phillips-Perron tests accept the hypothesis of a unit root–stochastic trend—for  $\ln(A)$  and for  $\ln(V)$ . I therefore estimate the regression in first differences:

$$d \ln(A_t) = \underset{0.0007}{0.0023} - \underset{0.086}{0.292} d \ln(V_t) + \varepsilon_t.$$

The relationship between these two variables is negative and strongly significant. The correlation is strongly negative:

$$\text{corr}[d \ln(A), d \ln(V)] = -0.22.$$

Considering sub-samples of this sample gives similar results. I conclude that the two time series for the shocks identified through the usual framework are negatively correlated. This result is consistent with the finding of [Schmitt-Grohe and Uribe \(2011\)](#) that the N-shock and the relative price of investment are cointegrated.

## 2.2 Calibration

The other dimension where the misspecification is notable is that the model predicts counter-factual savings rates.<sup>7</sup> To assess this, the model is simulated with the shocks identified from the data under a fairly standard parametrization summarized in table [1](#). The two shocks evolve according to the following

---

<sup>7</sup>In this model the saving rate is equal to the investment rate: investment over GDP. Following [Ríos-Rull et al. \(2009\)](#), investment includes private and public investment in equipment and structures, as well as consumer durable goods. This definition is consistent with the model in which consumption is only non-durable.

processes:

$$V_t = \gamma_{0,v} \gamma_{1,v}^t V_{t-1}^{\rho_v} e^{\varepsilon_{v,t}}, \quad \rho_v \leq 1, \quad (6)$$

$$A_t = \gamma_{0,a} \gamma_{1,a}^t A_{t-1}^{\rho_a} e^{\varepsilon_{a,t}}, \quad \rho_a \leq 1. \quad (7)$$

where  $\varepsilon_{v,t}$  and  $\varepsilon_{a,t}$  are independently and identically distributed random variables with standard deviation  $\sigma_{\varepsilon_v}$  and  $\sigma_{\varepsilon_a}$ .  $\gamma_{0,v}, \gamma_{1,v}, \gamma_{0,a}, \gamma_{1,a}$  are positive constants.  $\rho_a$  and  $\rho_v$  are restricted to being equal to one as suggested by the unit-root tests.  $\gamma_{0,a}, \gamma_{0,v}, \gamma_{1,a}, \gamma_{1,v}, \sigma_{\varepsilon_a}, \sigma_{\varepsilon_v}$  are estimated by running OLS regressions on the logs of the shocks.

The remaining parameters of the model are  $\beta, \alpha, \delta, \xi, \nu$ .  $\delta$  is equal to 0.014, the average depreciation rate of total capital calculated by [Cummins and Violante \(2002\)](#). The discount factor  $\beta$  is equal to 0.99. This parametrization implies an average capital-output ratio of 10.2, an investment-output ratio of 0.26, and an interest rate of 3.5%.

It remains to calibrate the parameters of the supply of labor; the critical one is  $\nu$ , which represents the Frisch elasticity. As pointed out by [King and Rebelo \(1999\)](#) among others, how much of the business cycle can be explained by technology shocks depends crucially on this parameter. Micro estimates suggest a small number: a recent survey of the micro evidence by [Chetty et al. \(2011\)](#) on the Frisch elasticity points to a value of 0.5 on the intensive margin and of 0.25 on the extensive margin. Macro studies point to a larger role of the extensive margin, which theoretically can lead  $\nu$  up to  $\infty$  even when the intensive margin is zero; see [Rogerson \(1988\)](#) and [Hansen \(1985\)](#). [Prescott \(2004\)](#) considers a value of approximately 3. [Ríos-Rull et al. \(2009\)](#) estimate the same model as in this section using Bayesian techniques and find posterior means between  $\nu = 0.12$  and  $\nu = 1.56$ , depending on the variables and the shocks included in the estimation.

In the context of the present application, which aims at measuring the extent to which the model replicates some empirical observations, in particular the observed saving rates, it seems instructive to consider a relatively high level of Frisch elasticity, to give the model the best chance of matching the data. Values of 0.75, 1.5 and 3 are considered.

Finally,  $\xi$  is chosen so that the average number of market hours is 0.33.

Table 1: Summary of Parametrization

Parameter	Moment to Match	Value
$\beta$	interest rate	0.99
$\delta$	direct measurement by <a href="#">Cummins and Violante (2002)</a>	0.014
$\alpha$	labor share	0.36
$\nu$	micro and macro evidence on Frisch elasticity	0.75, 1.5, 3
$\xi$	average market hours	11.97, 5.64, 3.87
$\gamma_{0,a}$	OLS	1.003
$\gamma_{0,v}$	OLS	1.004
$\gamma_{1,a}$	OLS	1
$\gamma_{1,v}$	OLS	1
$\rho_a$	ADF and PP tests	1
$\rho_v$	ADF and PP tests	1
$\sigma_{\varepsilon_a}$	OLS	0.0069
$\sigma_{\varepsilon_v}$	OLS	0.0051

### 2.3 Saving Rate

To compare the saving rate of the model with that in the data, the model is simulated with the time series of innovations  $\varepsilon_{a,t}$ ,  $\varepsilon_{v,t}$  identified from the data and initial conditions for  $A_0, V_0$  and  $k_0$ , all coming from the data. To avoid dependence on the initial conditions, the model is compared to the data from 1960 III (the 50th period of simulation).

The upshot of this experiment is that, although the model performs reasonably well according to the most common statistics used to evaluate it – standard deviations and covariances presented in tables 2 and 3 – the saving rate is very poor. Let  $\hat{s}$  be the time series predicted by the model, and  $s$  the time series realized. The two time series are so different from one another, that the  $R^2 = 1 - \text{var}(\hat{s} - s)/\text{var}(s)$  is even negative:  $R^2 = -0.036$  when  $\nu = 3$ ,  $R^2 = -0.013$  when  $\nu = 1.5$  and  $R^2 = 0.004$  when  $\nu = 0.75$ .

This shortcoming is not easily captured by simply looking at the correlations and standard deviations in tables 2 and 3. It is notable, however, how the model over-predicts the volatility of consumption and under-predicts that of investment. Also, consumption is too correlated to output, while investment

is less correlated than in the data. Similarly to other RBC models, the major shortcoming notable from these tables is that the model under-predicts the volatility of hours.

Table 2: Standard deviations

	Output	Consumption	Investment	Hours
Data	1.57	0.67	5.21	1.88
Model				
$\nu = 3$	1.03	0.72	2.73	0.47
$\nu = 1.5$	0.97	0.72	2.47	0.34
$\nu = 0.75$	0.92	0.72	2.25	0.22

Table 3: Correlation with output

	Output	Consumption	Investment	Hours
Data	1	0.40	0.95	0.87
Model				
$\nu = 3$	1	0.76	0.88	0.74
$\nu = 1.5$	1	0.78	0.86	0.69
$\nu = 0.75$	1	0.80	0.84	0.64

The two facts highlighted – the negative correlation between the shocks and the counterfactual saving rate – are taken as a sign of misspecification in the model. Alternatively, one could argue that the saving rate may move for other non technological shocks, not considered here (preference and government spending shocks are considered in Section 5.1). That notwithstanding, the poor performance highlighted calls for an investigation of this issue and the next subsection interprets the negative correlation between the shocks and the bad fit in the saving rate as being suggestive of a concave transformation frontier.

## 2.4 The Case for Curvature in the Transformation Frontier

To be consistent with the model, which can be expressed in recursive form with state variables  $A_t, V_t, k_t$ , assume that the true price equation takes the general form

$$p_t = p(A_t, V_t, k_t) \quad (8)$$

and let the total production measured in consumption units be

$$y_t = y(A_t, V_t, k_t). \quad (9)$$

Considering instead the price equation (3) and the aggregate resource constraint (2) would wrongly impute all the increase (decrease) in the relative price to a decrease (increase) in  $V_t$ , and all the variation in production not explained by the inputs, to  $A_t$ . If instead  $\frac{\partial p}{\partial A_t} > 0$ , increases in  $p_t$  may be due to increases in  $A_t$ , and when this happens  $y_t$  also increases through  $A_t$ . With the misspecified policy functions, the increase in the price would be attributed to a decrease in  $V_t$ , while instead only an increase in  $A_t$  occurred. This leads to the negative correlation between  $A_t$  and  $V_t$ , which is not a pure negative correlation between the two shocks, but is due to the misspecification of the model.

The misspecification also leads to counter-factual saving rates: when there is an increase in  $A_t$ , according to the true policy function (8)  $p_t$  grows. When this happens, the original model identifies a decrease in  $V_t$ . Because the productivity of investments decreased, the saving rate predicted by the model decreases. If, on the contrary, no I-shock occurred, the increase in  $A_t$  would imply an increase in the saving rate. Therefore, a misspecified price equation can lead to counter-factual saving rates.

It follows from these considerations that a model used to measure these shocks for the business cycle should be specified in a way such that it matches as closely as possible the saving rate time series and it identifies orthogonal, or at least not so strongly dependent, time series for the shocks.<sup>8</sup> These are the

---

<sup>8</sup>Some dependence may be justified by the fact that N- and I-shocks could stand for

two facts that will be targeted in the specification and calibration of the model presented in the next section.

### 3 The Modified Framework

Consider the following modification to the model: the representative firm's revenues are

$$y_t = A_t k_t^a n_t^{1-a} (1 - s_t)^{1-\rho} + p_t V_t A_t k_t^a n_t^{1-a} s_t^{1-\rho}, \quad (10)$$

where  $\rho \in [0, 1)$ .  $s_t$  measures the share of inputs allocated to the production of investment goods.

Therefore,

$$c_t = A_t k_t^a n_t^{1-a} (1 - s_t)^{1-\rho} \quad (11)$$

and

$$k_{t+1} - k_t(1 - \delta) = V_t A_t k_t^a n_t^{1-a} s_t^{1-\rho}. \quad (12)$$

An alternative way to induce concavity is to assume a constant elasticity of substitution aggregation of consumption and investment goods (see [Fisher \(1997\)](#)). That way to introduce curvature is essentially equivalent to the one proposed here. The advantage of this specification is that it more closely parallels the analysis in [Section 2](#) and derives a price equation which is a direct function of the I-shock and the saving rate (equation [\(14\)](#)).

The firm can produce for both sectors and solves the following problem:

$$\max_{k, n, s \in [0, 1]} A_t k_t^a n_t^{1-a} (1 - s_t)^{1-\rho} + p_t V_t A_t k_t^a n_t^{1-a} s_t^{1-\rho} - wn - rp k. \quad (13)$$

When  $\rho > 0$ , the marginal productivity of consumption and of investment goods is decreasing and therefore the firm will choose to produce both types of goods even when  $pV$  differs from one. This very simple specification, capturing curvature in somewhat reduced form, has the advantage of being closely

---

a mixture of multi-factor productivity shocks in a multi-sector model. See [Guerrieri et al. \(2009\)](#) for a careful mapping of the one sector model with N- and I-shocks into a fully fledged multi-sector model.

related to the original framework from which it departs, thereby being able to isolate the role of curvature from any other possible change that can be made. In particular, this technology preserves the assumption of constant returns to scale on capital and labor, so the problem remains consistent with perfect competition. Furthermore, the desirable growth properties of the linear framework are preserved, see [Greenwood et al. \(1997\)](#).<sup>9</sup>

The equilibrium conditions that correspond to a competitive equilibrium are reported in [Appendix A.2](#) and are essentially unchanged with respect to the usual framework, except for the resource constraints above and for the price equation, which comes from the optimal choice of  $s_t$ :

$$p_t V_t = \frac{(1 - s_t)^{-\rho}}{s_t^{-\rho}}. \quad (14)$$

This price equation shows that the change in the relative price is not only due to a change in  $V_t$ , but also depends on the change in  $s_t$ , i.e. on the change in the relative demand for the two goods. This in turns depends on both the shocks and on capital. The reason for this is that the production possibility frontier is concave as illustrated in [Figure 1](#).

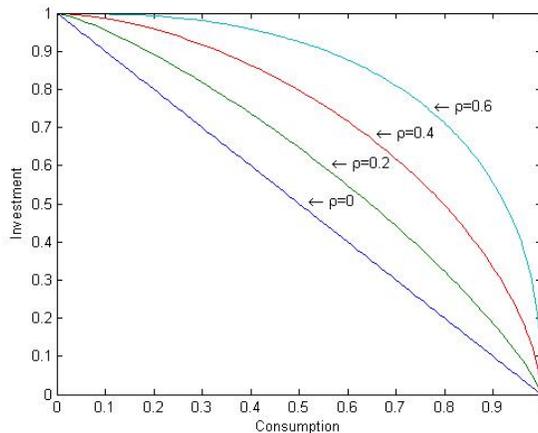


Figure 1: Production possibility frontier

---

<sup>9</sup>The modification introduced leaves the balanced growth path unchanged and therefore it maintains the same growth implications of the original framework as shown in [Appendix 2](#), section [A.2](#).

When  $\rho = 0$  the price equation (and the whole model) boils down to the usual framework. The next section pins down  $\rho$ . Before doing that, it is already possible to relate to the news shock literature. [Beaudry and Portier \(2007\)](#) define a model consistent with expectation driven business cycles when a change in expectations of the future technology (N- and I-shocks in the context of this paper) generates a positive comovement between consumption, investment and labor, holding current technology and preferences constant. Their Proposition 2 offers a necessary condition for an economy to exhibit expectation driven business cycles. This condition, in the context of this model, boils down to the following:

$$\frac{\rho}{1-\rho} \frac{p_t s_t (1-\alpha)}{n_t} \left( \frac{1}{1-s_t} + s_t^{\rho(1-\rho)} \right) > 0. \quad (15)$$

With  $p_t$  and  $s_t$  positive, this condition is only satisfied with  $\rho$  strictly positive, i.e. with a concave frontier. To appreciate this result, it is important to notice that [Beaudry and Portier \(2007\)](#) show that expectation driven business cycles *cannot* arise with capital adjustment costs or capital utilization when the frontier is linear. Indeed, [Beaudry and Portier \(2007\)](#) suggest two examples that generate expectation driven business cycles: a multi-sector model with cost complementarities and a one sector model with a distribution system. These set-ups essentially introduce curvature in the frontier between consumption and investment.

### 3.1 Estimating $\rho$

As mentioned, two strategies are employed. The first is to pick  $\rho$  such that the shocks identified are uncorrelated. Similarly to the original model, the shocks can be identified from the production equation(10) and the price equation (14) as

$$V = \left( \frac{1-s}{s} \right)^{-\rho} \frac{1}{p} \quad (16)$$

$$A = \frac{y}{k_t^\alpha n_t^{1-\alpha} [(1-s)^{1-\rho} + pV s^{1-\rho}]}. \quad (17)$$

As becomes clear from observing the two equations above, to identify the shocks it is first necessary to identify  $s$ . From the resource constraints (11) and (12)

it follows that

$$\frac{(1-s)^{1-\rho}}{pVs^{1-\rho}} = \frac{c}{y-c}. \quad (18)$$

Substituting into equation (14) one gets

$$s = 1 - \frac{c}{y}. \quad (19)$$

$s$  is the saving rate which can be taken from the data.

With  $s, k, y, p$  and  $n$  at hand, at each  $\rho$  there are corresponding time series for  $A$  and  $V$  through (16) and (17) and a correlation  $\text{corr}(\varepsilon_a, \varepsilon_v)$  from estimation of the processes (6) and (7). Figure 2 shows this correlation as a function of  $\rho$ .

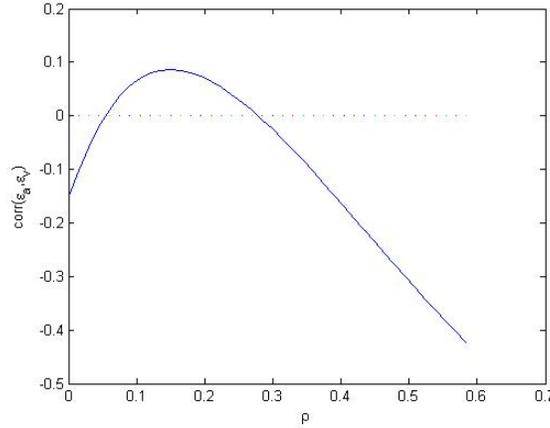


Figure 2: Correlation between the innovation of the shocks as a function of  $\rho$ .

As can be observed, the correlation is concave, and it crosses zero twice. It should be clear that, starting with a linear frontier ( $\rho = 0$ ), an increase in  $\rho$  reduces the correlation for the reasons given in section 2. The figure also shows that for  $\rho$  sufficiently high the correlation is decreasing in  $\rho$ . This can be rationalized as follows: after a positive I-shock, inter-temporal optimization calls for an increase in the saving rate  $s$ . The increase in  $s$  implies that the marginal productivity of consumption goods  $(1-\rho)A_t k_t^\alpha n_t^{1-\alpha}(1-s_t)^{-\rho}$  increases. The marginal productivity of investments, measured in consumption goods  $p_t V_t (1-\rho)A_t k_t^\alpha n_t^{1-\alpha} s_t^{-\rho}$  also has to increase, since the two marginal productivities must be equal in equilibrium. This calls for an increase in  $pV$ . Compared to the original framework, the price reacts less to a change in the

investment shock, making the product  $pV$  procyclical. Unlike what happens in the original framework, the fact that  $pV$  increases even after an investment shock, makes aggregate productivity increase. With too much curvature, this effect may be exacerbated, total output is predicted to increase more than in the data and a negative N-shock is identified. Thus, the correlation between the two shocks is negative if  $\rho$  is too high.

The second strategy to pin down  $\rho$  is to maximize the  $R^2$  of the saving rate predicted by the model given the shocks identified.<sup>10</sup> This is done through a grid search over  $\rho$  and for each value of  $\rho$ , by doing the following: 1. given the other parameters, back out the two shocks' time series through (16) and (17); 2. Estimate the parameters of the shocks' processes. 3. Solve the model.<sup>11</sup> 4. Simulate given the shocks identified, and compute the  $R^2$  after discarding the first 50 observations.

Figure 3 plots the  $R^2$  as a function of  $\rho$ .

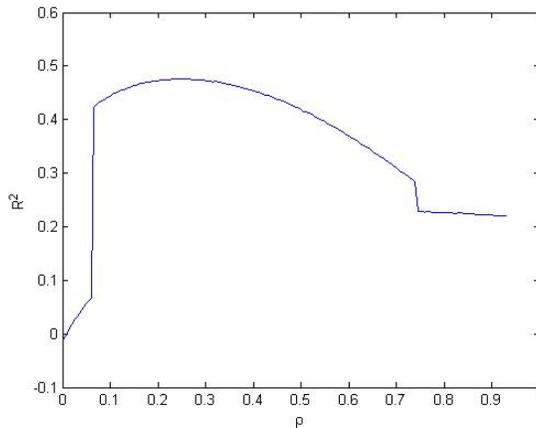


Figure 3:  $R^2$  as a function of  $\rho$ .

There is a kink when  $\rho$  is approximately 0.06. At that point the N-shock becomes stationary. This leads to a much higher portion of the variance being explained. With Frisch elasticity  $\nu = 1.5$ , the value of  $\rho$  that gives the highest

<sup>10</sup>This is equivalent to minimizing the squared sum of residuals  $s - \hat{s}$ .

<sup>11</sup>As explained below, with higher values of  $\rho$  the N-shock is stationary. When this is the case, the model is solved assuming a stationary process for the N-shock and maintaining a non-stationary process for the I-shock.

$R^2$  is  $\rho = 0.243$ .  $R^2$  of 0.476 is a substantial increase in the portion of variance of the saving rate explained by this model compared to the original framework, where the variance explained is essentially zero. With  $\nu = 0.75$ , this procedure leads to  $\rho = .233$  with  $R^2 = .470$ . With  $\nu = 3$ ,  $\rho = .252$  and  $R^2 = .481$ .

Strikingly, these estimates for  $\rho$  are very close to the highest of the two values obtained with the other procedure. In fact the properties of the shock processes are essentially unchanged when  $\rho$  is found by maximizing the saving rate or with the highest value obtained through the correlation procedure. This suggests that among the two values estimated through the correlation procedure, the highest value may be favored. To remove any doubt, a GMM procedure is run where the moments above – the correlation between the residuals  $\text{corr}(\varepsilon_a, \varepsilon_v)$  and the sum of squares  $(s - \hat{s})$  – are combined. Not surprisingly, this procedure gives a value between the one that maximizes the  $R^2$  and the highest value obtained through the correlation procedure. These estimations are summarized in table 4. The table also reports the standard deviation of the parameter estimated and the p-value of the  $J$  test for over-identification, which does not reject the null that the model is correctly specified. Given the asymptotic normality of the GMM estimator, standard errors suggest that  $\rho$  is significantly larger than zero. These results are not very sensitive to the stand taken on  $\nu$ .

Table 4: Curvature Parameter  $\rho$

Method	Estimate	St.Dev	J test (p-value)
1 <sup>st</sup> strategy	0.279	-	-
$\nu = 0.75$			
2 <sup>nd</sup> strategy	0.233	-	-
GMM	0.266	0.065	0.424
$\nu = 1.5$			
2 <sup>nd</sup> strategy	0.243	-	-
GMM	0.265	0.047	0.428
$\nu = 3$			
2 <sup>nd</sup> strategy	0.252	-	-
GMM	0.267	0.049	0.432

Thus,  $\nu = 1.5$  is considered hereafter unless otherwise specified.

Figure 4 compares the saving rate from the data with those of the model with  $\rho = 0.265$  and  $\rho = 0$ . As is evident, curvature induces a substantial improvement.

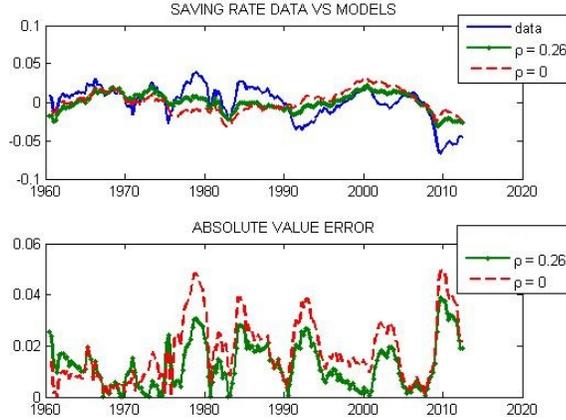


Figure 4: Saving rate: deviation from the mean

## 4 Results

Given the value of  $\rho$  estimated with GMM, the parameter values for the shock processes are summarized in table 4. A first result is that while the investment shock remains a unit root, as happens when  $\rho = 0$ , the neutral one is now trend-stationary. ADF tests, with various lags, reject the hypothesis of a unit root for the neutral shock with p-values that range between 1% and 9%. The Philip-Perron test rejects the hypothesis of a unit root with p-values always below 5%.<sup>12</sup>

<sup>12</sup>Whether the business cycle is about stationary fluctuations around a deterministic trend, or is due to a stochastic trend has been debated since the paper by Nelson and Plosser (1982). This is important because the response to permanent shocks is typically weaker than the reaction to a transitory shock. There is a simple intuition for this result: when there is a permanent shock, productivity grows but so does expected wealth. Therefore, the expected marginal utility of consumption decreases, lowering the boost in the saving rate and in the labour supply. In most previous studies, the two shocks have either been considered both stationary, as for instance in Smets and Wouters (2007), or both unit roots, as in Fisher (2006) and Schmitt-Grohe and Uribe (2011) among others, or the N-shock was assumed to

As with the baseline framework, the process for the N-Shock does not show a significant trend: all the growth is captured by the I-Shock. Appendix [A.3](#) derives the equivalent stationary conditions when there is a trend-stochastic shock and a stationary one. The transformed stationary model proves the existence of a Balanced Growth Path and allows for a recursive formulation. From this it becomes clear that the model has the same long-run implications as the original framework: the expected growth rates of all the variables are unchanged.

Table 5: Other Parameter Values

$\gamma_{0,a}$	$\gamma_{1,a}$	$\rho_a$	$\sigma_{\varepsilon_a}$	$\gamma_{0,v}$	$\gamma_{1,v}$	$\rho_v$	$\sigma_{\varepsilon_v}$
1.017	1.000	0.983	0.006	1.000	1.005	1.000	0.012

Tables [6](#) and [7](#) report standard deviations and correlations with output: the volatility of consumption—too high in the linear case—decreases, that of investment—too low in the linear case—increases. Correlations also move in the right direction.

Table 6: Standard deviations ( $\nu = 1.5$ )

	Output	Consumption	Investment	Hours
Data	1.57	0.67	5.21	1.88
Model				
$\rho = 0$	0.97	0.72	2.47	0.34
$\rho = 0.265$	0.92	0.64	2.54	0.36

It is possible to revisit the age-old question originated by [Kydland and Prescott \(1982\)](#), of how much of the business cycle is accounted for by technology shocks. With curvature, technology shocks account for 59% of the aggregate fluctuations in output, slightly less than with a linear frontier. However, the volatility of hours and their correlation with output are slightly higher.

---

be a unit root and the I-shock stationary ([Justiniano et al. \(2010\)](#)).

Table 7: Correlation with output ( $\nu = 1.5$ )

	Output	Consumption	Investment	Hours
Data	1	0.40	0.95	0.86
Model				
$\rho = 0$	1	0.78	0.86	0.69
$\rho = 0.265$	1	0.75	0.87	0.74

The volatility of consumption, which was too high with a linear frontier, is now lower. That of investment, too low with a linear frontier, is now higher. These results, closely related to the better fit of the saving rate, are best understood in the light of the impulse response functions to the two shocks.

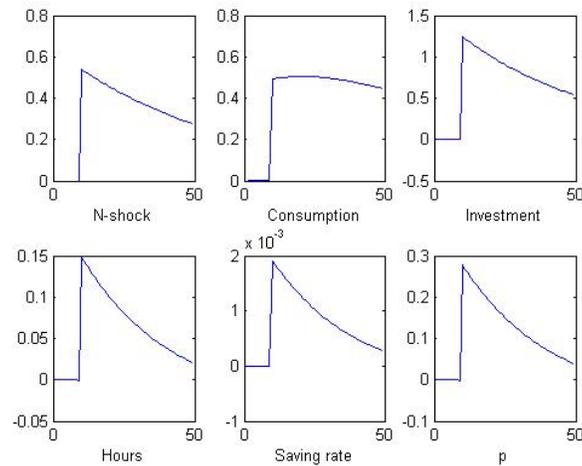


Figure 5: Impulse response function to an N-shock.

All variables other than the saving rate are expressed in percentage change, from steady state. The saving rate panel shows the change in level from steady state.

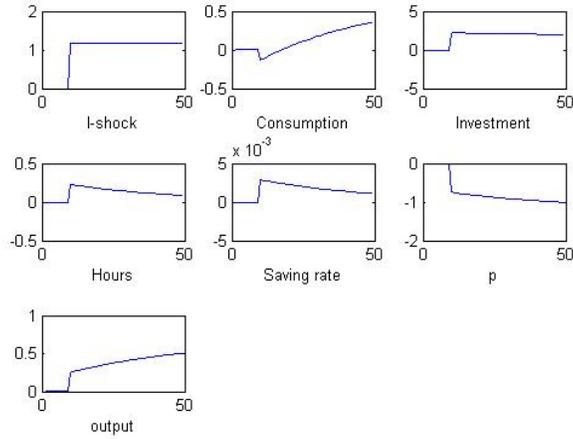


Figure 6: Impulse response function to an I-shock.

All variables other than the saving rate are expressed in percentage change, from steady state. The saving rate panel shows the change in level from steady state.

1. As shown in Figure 5, after a positive neutral shock, households want to increase the investment rate in order to smooth consumption. With a concave frontier, firms are reluctant to accommodate this excess demand for investment goods and the price has to increase to induce them to adjust the supply. This highlights the fact that the change in the relative price of goods is not all due to the I-shock and how misleading it could be to identify the investment shock in the usual way.

The fact that  $p$  increases after an N-shock implies that consumption responds more to the shock relative to the linear framework; the increase in the relative price induces agents to increase consumption, preventing the saving rate from increasing as much as in the linear framework, where  $p$  does not depend on the N-shock. This is a feature typical of two-good models with imperfect input reallocation, which turn out to imply a high equity premium as has been shown by [Christiano et al. \(2001\)](#).

2. As shown in Figure 6, after an I-shock consumption decreases. However, the decrease is reduced by the smaller (compared to the linear framework) decrease in the price that follows the investment shock. Thus  $pV$

increases, contributing to the increase in GDP measured in consumption goods. Given this increase in GDP, it is possible to increase the saving rate without an abrupt decrease in consumption.

The extent to which this dynamic is an improvement relative to the linear framework can be appreciated by comparing it with the impulse responses from the linear framework, once the negative correlation is taken into account. As shown in Figure 7, after a positive I-shock output *decreases* for a prolonged period of time. This is because the I-shock also leads to a negative effect on the N-shock.<sup>13</sup> Furthermore, if the negative effect on the N-shock is large enough, an I-shock also induces a *decrease* in the saving rate and hours as shown in Figure 8. These figures highlight how the typical propagation of I-shocks in the model with a linear frontier is hard to rationalize and calls for a concave frontier. One reason why these effects have been overseen might be that typically the correlation between the shocks is ignored. One exception is [Schmitt-Grohe and Uribe \(2011\)](#) who assume a co-integrated process for the two shocks.

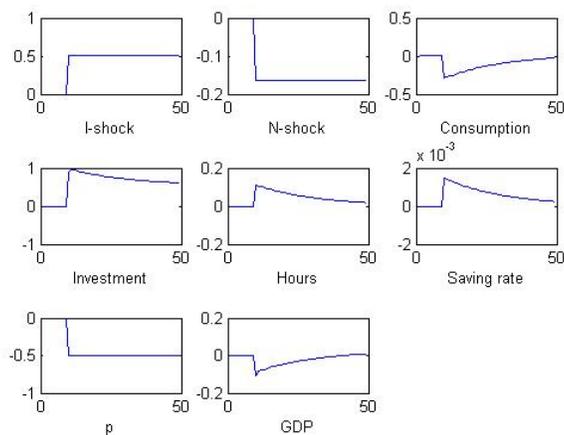


Figure 7: Impulse response function to an I-shock with  $\rho = 0$ . All variables other than the saving rate are expressed in percentage change, from steady state. The saving rate panel shows the change in level from steady state.

<sup>13</sup>This comes from a Choleski decomposition applied to the covariance matrix between the innovations, so that an innovation to the I-shock can affect the N-shock.

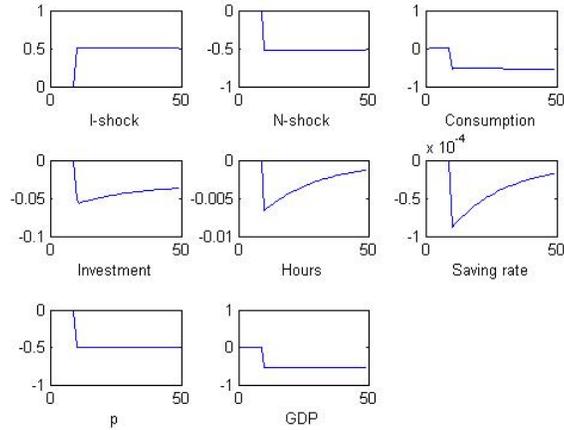


Figure 8: Impulse response function to an I-shock with  $\rho = 0$  and strong negative effect on N-shock.

All variables other than the saving rate are expressed in percentage change, from steady state. The saving rate panel shows the change in level from steady state.

Finally, it is possible to get a sense of the relative importance of the two shocks in the model with curvature. Running the model with either only N-shocks or I-shocks, it is found that most of the variability in output comes from the N-shock, while most of that in hours comes from the I-shock. In particular, running the model with N-shocks only, 66% of the standard deviation of output is explained but only 6% of that of hours.<sup>14</sup> Running the model only with I-shocks, 20% of the standard deviation of output is explained and 18% of that of hours. The fact that hours are more sensitive to the I-shock has also been found by [Ríos-Rull et al. \(2009\)](#).

## 5 Other Changes in Specification

How does curvature in the frontier substitute for other possible changes in specification that many authors have investigated? I extend the model to include capital adjustment costs, habits in consumption and capital utilization.

<sup>14</sup>In the linear case N-shocks account for 56% of the standard deviation of output and 7% of that of hours. Curvature amplifies the effects of N-shocks.

As intuition suggests, they could affect saving decisions and the identification of the relative price.

The household is faced with the following capital accumulation equation:

$$k_{t+1} = k_t(1 - \delta) + \psi\left(\frac{i_t}{k_t}\right)k_t, \quad (20)$$

where  $i_t$  is investment and  $\psi(\cdot)$  is an increasing and concave capital adjustment cost function.

The budget constraint is

$$c_t + p_{k,t}i_t = w_t n_t + k_t R_t. \quad (21)$$

The production of consumption is

$$c_t = A_t(u_t k_t)^a n_t^{1-a} (1 - s_t)^{1-\rho} \quad (22)$$

where  $u_t$  is the intensity with which capital is used. The production of the other sector is

$$i_t + a(u_t)k_t = V_t A_t(u_t k_t)^a n_t^{1-a} s_t^{1-\rho} \quad (23)$$

where  $a(u_t)k_t$  is the input consumed in the production process depending on  $u_t$ . Following the literature, I assume  $u_t = 1$  in steady state,  $a(1) = 0$ , and define  $\chi = \frac{a''(1)}{a'(1)}$ : up to a first order Taylor approximation of the equilibrium conditions, this parameter is the only one that has to be pinned down to determine the cost function.

The firm can produce for both sectors and solves the following problem:

$$\max_{k,n,u,s \in [0,1]} A_t k_t^a n_t^{1-a} (1 - s_t)^{1-\rho} + p_t V_t A_t k_t^a n_t^{1-a} s_t^{1-\rho} - wn - rpk - a(u_t)pk. \quad (24)$$

The utility function of the household is changed to

$$\max_{\{c_t, k_{t+1}, n_t\}} E_0 \left[ \sum_{t=0}^{\infty} \beta^t \left( \log(c_t - bc_{t-1}) - \chi \frac{n_t^{1+1/\nu}}{1 + 1/\nu} \right) \right]$$

with  $c_{-1}$  given.

The equilibrium conditions are reported in Appendix [A.2](#). The first finding is that the relative price equation that identifies I-shocks—equation [\(30\)](#)—is not affected by either of these frictions.<sup>15</sup>

<sup>15</sup>Assuming that adjustment costs are borne by the firms would be equivalent.

Following [Jermann \(1998\)](#),

$$\psi\left(\frac{i_t}{k_t}\right) = \frac{a1}{1-\zeta}\left(\frac{i_t}{k_t}\right)^{1-\zeta} + a2, \quad \zeta > 0, \quad (25)$$

with  $\zeta = 0.23$ . This determines the elasticity of investments to Tobin Q.  $a1$  and  $a2$  are such that the model has the same deterministic steady state as in the case with no adjustment costs.

In a model very close to the present one with a linear frontier, [Justiniano et al. \(2010\)](#) estimate  $b = 0.86$  and  $\chi = 5.4$ .

Putting  $\rho = 0$  and simulating the model with the shocks identified gives an  $R^2$  of 0.30. With all these frictions, this statistic is clearly worse than with a concave transformation frontier alone. Even though capital utilization makes the Solow residual partly endogenous, the correlation between the shocks' innovations remains strongly negative:  $-0.194$ .

Tables 8 and 9 compare the business cycle statistics of the model with adjustment costs to that with a concave transformation frontier. The most noticeable result is a negative correlation between hours and output.<sup>16</sup> This is intuitive: given that with adjustment costs and habits capital and consumption have to be kept smooth, when firms increase intensity, hours may decline. Indeed, if habits are reduced to  $b = 0.2$ , the correlation between output and hours becomes 0.69. The fit of the saving rate increases to  $R^2 = 0.34$ . However, the volatility of consumption increases and the one of hours decreases; both facts are counterfactual. In conclusion, there is a tension between the goal of smoothing consumption and having hours volatile and co-moving with output. Curvature helps to attenuate this problem as shown in the last row of the tables: the volatility of hours increases and the one of consumption decreases.

From these findings, it seems clear that the alternative frictions considered, while improving the data generating process, do not provide a substitute to a concave frontier for the dimensions this paper is concerned with.<sup>17</sup>

<sup>16</sup>This result is also mentioned in [Christiano et al. \(2001\)](#) for the divisible labor case.

<sup>17</sup>Another friction commonly used in the DSGE literature is to assume sticky prices and monetary policy. For how sticky prices are modeled in the literature—a sticky aggregate price (see for instance [Justiniano et al. \(2011\)](#))—it does not affect the relative price between

Table 8: Standard deviations ( $\nu = 1.5$ )

	Output	Consumption	Investment	Hours
Data	1.57	0.67	5.20	1.88
Models				
$\rho = 0.265$	0.92	0.64	2.54	0.36
Other frictions with $b = 0.86$	0.90	0.41	2.79	0.30
Other frictions with $b = 0.2$	1.10	0.86	2.09	0.19
$\rho = 0.265$ and other frictions with $b = 0.2$	0.95	0.79	1.94	0.23

Table 9: Correlation with output ( $\nu = 1.5$ )

	Output	Consumption	Investment	Hours
Data	1	0.40	0.95	0.86
Models				
$\rho = 0.265$	1	0.75	0.87	0.74
Other frictions with $b = 0.86$	1	0.75	0.95	-0.38
Other frictions with $b = 0.2$	1	0.95	0.93	0.69
$\rho = 0.265$ and other frictions with $b = 0.2$	1	0.91	0.86	0.48

## 5.1 Bayesian Estimation

While the previous section used calibration to highlight the role of curvature relative to other frictions commonly used in the literature, practitioners that may want to include a concave frontier in their models together with other frictions might find it convenient to estimate these parameters all together with full information criteria as is now common in the literature. Even though likelihood based estimators without an adequate prior sometimes lead to incredible parameter values (see [An and Schorfheide \(2007\)](#)) the exercise may also be

---

consumption and investment and the identification of the I-shock. Alternatively, one could model sticky prices for the consumption and investment goods separately. This way the relative price would be detached from the I-shock: with the relative price fix, one could then pick the I-shock to improve the saving rate, but curvature would be loosely identified. For this reason, sticky prices are not included in the model.

taken as a further test for the presence of curvature.

I estimate  $\rho$ , and the parameters listed in Table 10 using Bayesian methods. The other parameters not listed in the table ( $\beta$ ,  $\alpha$  and  $\delta$ ) are calibrated as in the previous sections. The investment shock is assumed to have a unit root. The observables are the growth rates of consumption, output, hours and the relative price, adequately transformed to have an exact counterpart in the log-linear approximation of the model (see Appendix A.3).

To see if the Bayesian criterium favors a concave specification, the prior for  $\rho$  is uniform between zero and one. Priors for  $\nu$ ,  $\chi$ ,  $b$  and the variance of the shocks' innovations are the ones used by Justiniano et al. (2011). The prior for the other parameters, which do not have an exact counterpart in Justiniano et al. (2011), are centered to those calibrated in the previous section. To fully match the data, an *i.i.d* preference shock which shows up in the first order condition for leisure, and an *i.i.d* government spending shock that adds to the demand for consumption goods are introduced. Table 10 summarizes the prior and the estimated parameters.

I also allow for measurement errors for the four observables, and use the Kalman filter.<sup>18</sup>

$\rho$  is positive. Even though lower than what is found in the previous sections by matching moments, this result reinforces the claim that the frontier should be concave because the identification of curvature relies on the whole covariance structure of the observables, not just on the dynamics of the saving rate which—it has been argued—calls for a concave frontier. Furthermore, since capital is not included in the vector of observables, the N-shock is not identified as a Solow residual, thus the correlation among the shocks may not help identifying curvature the same way it does in the previous sections. A further reason why this result reinforces the claim of a concave frontier is that preference and government spending shocks provide alternative sources of variation for the saving rate, thereby allowing the model to match the saving rate time series even without curvature. Indeed, the prior has been chosen uniform to show

---

<sup>18</sup>The prior for the variances in the measurement errors are inverse Gamma with parameters (0.1, 1).

Table 10: Prior densities and posterior estimates

Parameter	Description	Prior			Posterior	
		Density	Para(1)	Para(2)	Estimate	[5 – 95]
$\rho$	Curvature	Uniform	0	1	0.104	[0.045 – 0.179]
$1/\nu$	Inverse Frisch elasticity	Gamma	2	0.75	2.838	[1.680 – 3.183]
$\chi$	Elasticity capital utilization cost	Gamma	5	1	3.749	[3.027 – 5.887]
$\zeta$	Investment adjustment cost	Beta	0.23	0.2	0.856	[0.465 – 0.936]
b	Habit	Beta	0.5	0.1	0.612	[0.522 – 0.692]
$\rho_a$	Persistence N-shock	Beta	0.9	0.05	0.988	[0.970 – 0.994]
$\gamma_{1,a}$	Trend N-shock	Gamma	1	0.01	1.000	[1.000 – 1.000]
$\gamma_{0,a}$	Constant N-shock	Gamma	1.02	0.01	1.024	[1.016 – 1.044]
$\sigma_a$	Std N-shock	Inv Gamma	0.1	1	0.026	[0.021 – 0.044]
$\gamma_v$	Growth rate I-shock	Gamma	1.005	0.01	1.015	[1.001 – 1.029]
$\sigma_v$	Std I-shock	Inv Gamma	0.1	1	0.009	[0.008 – 0.011]
$\sigma_\varphi$	Std preference	Inv Gamma	0.1	1	0.025	[0.017 – 0.027]
$\sigma_g$	Std Gov. spending	Inv Gamma	0.1	1	0.043	[0.034 – 0.095]

Notes: Para(1) and Para(2) represent the lower and upper-bound of the uniform distribution. For all other distributions, they are the mean and standard deviation. Posterior percentiles are from 2 chains of 80,000 draws generated using a Random Walk Metropolis algorithm. The initial 30,000 draws are discarded.

that the data call for a concave frontier even when the moments used in the previous sections are ignored or implicit in the likelihood function. However, practitioners that want to put more weight on the moments highlighted in this paper, may use a prior centered to the value of  $\rho$  found in the previous sections.

The business cycle implications of the estimated parameters are essentially a convex combination of the results of the previous sections and are thus omitted.

## 6 Conclusions

This paper shows that the basic framework adopted by the DSGE literature predicts counter-factual saving rates when simulated with the shocks identified from the data. This result depends on the fact that the model identifies negatively correlated neutral and investment shocks. I argue that this counter-factual observation emanates from the assumed linearity of the transformation frontier between consumption and investment goods.

A simple extension of the original framework is developed that allows for the transformation frontier to be concave while preserving the long run properties of the linear framework, broadly consistent with evidence. With a concave frontier, the model improves dramatically in its prediction of the saving rate, a dimension that cannot be significantly improved with alternative mechanisms such as capital adjustment costs, habit persistence, and capital utilization. Furthermore, the paper shows that curvature has the potential of generating expectation driven business cycles.

Given the promising results and the simple modeling approach, introducing curvature in the transformation frontier into the large-scale models adopted by the DSGE literature may be a fruitful avenue to pursue. Another interesting direction for future research is to investigate the underlying factors that lead to an aggregate concave frontier.

Finally, a better understanding of the saving rate dynamics can be useful to shed light on other economic questions. For example, ([Mennuni \(2013\)](#)) uses this theory to isolate periods in which changes in the saving rate are not due to technological shocks and to test the paradox of thrift hypothesis.

## References

An, S. and F. Schorfheide (2007). Bayesian Analysis of DSGE Models. *Econometric Reviews* 26(2-4), 113–172.

Beaudry, P. and F. Portier (2007, July). When can changes in expectations

- cause business cycle fluctuations in neo-classical settings? *Journal of Economic Theory* 135(1), 458–477.
- Chetty, R., A. Guren, D. S. Manoli, and A. Weber (2011, January). Does indivisible labor explain the difference between micro and macro elasticities? A meta-analysis of extensive margin elasticities. Working Paper 16729, National Bureau of Economic Research.
- Christiano, L. J., M. Boldrin, and J. D. M. Fisher (2001, March). Habit persistence, asset returns, and the business cycle. *American Economic Review* 91(1), 149–166.
- Cummins, J. and G. Violante (2002, April). Investment-specific technical change in the U.S. (1947-2000): Measurement and macroeconomic consequences. *Review of Economic Dynamics* 5(2), 243–284.
- Fisher, J. (2006, June). The dynamic effects of neutral and investment-specific technology shocks. *Journal of Political Economy* 114(3), 413–451.
- Fisher, J. D. M. (1997, August). Relative prices, complementarities and comovement among components of aggregate expenditures. *Journal of Monetary Economics* 39(3), 449–474.
- Floetotto, M., N. Jaimovich, and S. Pruitt (2009). Markup variation and endogenous fluctuations in the price of investment goods. Technical report.
- Gordon, R. J. (1990, August). The measurement of durable goods prices. (gord90-1).
- Greenwood, J., Z. Hercowitz, and G. Huffman (1988, June). Investment, capacity utilization and the business cycle. *American Economic Review* 78.
- Greenwood, J., Z. Hercowitz, and P. Krusell (1997). Long-run implications of investment-specific technological change. *American Economic Review* 87(3).
- Greenwood, J., Z. Hercowitz, and P. Krusell (2000). The role of investment specific technological change in the business cycle. *European Economic Review*.

- Guerrieri, L., D. Henderson, and J. Kim (2009). Interpreting investment-specific technology shocks. *mimeo, Federal Reserve Board*.
- Hansen, G. D. (1985). Indivisible labor and the business cycle. *Journal of Monetary Economics* 16(3), 309 – 327.
- Huffman, G. and M. Wynne (1999). The role of intratemporal adjustment costs in a multisector economy. *Journal of Monetary Economics* 43(2).
- Jaimovich, N. and S. Rebelo (2009). Can news about the future drive the business cycle? *American Economic Review* 99(4).
- Jermann, U. J. (1998, April). Asset pricing in production economies. *Journal of Monetary Economics* 41(2), 257–275.
- Justiniano, A., G. Primiceri, and A. Tambalotti (2011, January). Investment shocks and the relative price of investment. *Review of Economic Dynamics* 14(1), 101–121.
- Justiniano, A. and G. E. Primiceri (2008, June). The Time-Varying Volatility of Macroeconomic Fluctuations. *American Economic Review* 98(3), 604–41.
- Justiniano, A., G. E. Primiceri, and A. Tambalotti (2010, March). Investment shocks and business cycles. *Journal of Monetary Economics* 57(2), 132–145.
- Kim, J. (2003, February). Functional equivalence between intertemporal and multisectoral investment adjustment costs. *Journal of Economic Dynamics and Control* 27(4), 533–549.
- King, R. G. and S. T. Rebelo (1999, June). Resuscitating real business cycles. In J. B. Taylor and M. Woodford (Eds.), *Handbook of Macroeconomics*, Volume 1 of *Handbook of Macroeconomics*, Chapter 14, pp. 927–1007. Elsevier.
- Kydland, F. E. and E. C. Prescott (1982). Time to build and aggregate fluctuations. *Econometrica* 50(6), pp. 1345–1370.
- Mennuni, A. (2013). A neoclassical approach to the paradox of thrift. *Mimeo*.

- Nelson, R. and C. Plosser (1982). Trends and random walks in macroeconomic time series. *Journal of Monetary Economics*.
- Plosser, C. I. (1989, Summer). Understanding Real Business Cycles. *Journal of Economic Perspectives* 3(3), 51–77.
- Prescott, E. C. (2004). Why do americans work so much more than europeans? *Quarterly Review* (Jul), 2–13.
- Rogerson, R. (1988). Indivisible labor, lotteries and equilibrium. *Journal of Monetary Economics* 21(1), 3 – 16.
- Ríos-Rull, J.-V., F. Schorfheide, C. Fuentes-Albero, M. Kryshko, and R. Santaeulàlia-Llopus (2009, September). Methods versus substance: Measuring the effects of technology shocks on hours. Working Paper 15375, National Bureau of Economic Research.
- Schmitt-Grohe, S. and M. Uribe (2011, January). Business cycles with a common trend in neutral and investment-specific productivity. *Review of Economic Dynamics* 14(1), 122–135.
- Smets, F. and R. Wouters (2007, June). Shocks and frictions in U.S. business cycles: A Bayesian DSGE approach. *American Economic Review* 97(3), 586–606.
- Valles, J. (1997, June). Aggregate investment in a business cycle model with adjustment costs. *Journal of Economic Dynamics and Control* 21(7), 1181–1198.

## A Appendix

### A.1 Data

The data-set extends the data-set of [Ríos-Rull et al. \(2009\)](#) to 2012 II. See their online appendix for the construction of a price index for consumption, a

quality-adjusted price index for investment, quality-adjusted investment and capital.

### A.1.1 Raw Data Series

*Bureau of Labor Statistics (BLS)*

Hours, ID PRS85006033

Civilian Noninstitutional Population, ID LNU00000000

*National Income and Product Accounts (NIPA-BEA)*

Nominal Gross National Product, Table 1.7.5

Price Indexes for Private Fixed Investment by Type, Table 5.3.4

Private Fixed Investment by Type, Table 5.3.5

Gross Domestic Product, Table 1.1.5

Government Consumption Expenditures and Gross Investment, Table 3.9.5

Personal Consumption Expenditures by Major Type of Product, Tables 2.3.5 (Nominal) and 2.3.3 (Quantity Index)

*Cummins and Violante (2002)*

Annual Quality-Adjusted Price Index for Investment in Equipment

Annual Quality-Adjusted Depreciation Rates for Total Capital

For Online Publication

## A.2 Balanced Growth Path with trend-stochastic shocks

The equilibrium conditions are

$$\lambda_t = \beta E_t \left\{ \frac{1}{c_{t+1}} R_{t+1} + \lambda_{t+1} \left( 1 - \delta + \psi \left( \frac{i_{t+1}}{k_{t+1}} \right) - \psi' \left( \frac{i_{t+1}}{k_{t+1}} \right) \frac{i_{t+1}}{k_{t+1}} \right) \right\} \quad (26)$$

$$\frac{1}{c_t} p_t = \lambda_t \psi' \left( \frac{i_t}{k_t} \right) \quad (27)$$

$$\frac{w_t}{c_t} = \xi n_t^{1/\nu} \quad (28)$$

$$k_{t+1} = (1 - \delta)k_t + \psi\left(\frac{i_t}{k_t}\right) k_t \quad (29)$$

$$p_t V_t = \frac{(1 - s_t)^{-\rho}}{s_t^{-\rho}} \quad (30)$$

$$i_t = V_t A_t k_t^\alpha n_t^{1-\alpha} s_t^{1-\rho} \quad (31)$$

$$c_t = A_t k_t^\alpha n_t^{1-\alpha} (1 - s_t)^{1-\rho} \quad (32)$$

$$w_t = (1 - \alpha) A_t k_t^\alpha n_t^{-\alpha} [(1 - s_t)^{1-\rho} + p_t V_t s_t^{1-\rho}] \quad (33)$$

$$R_t = \alpha A_t k_t^{\alpha-1} n_t^{1-\alpha} [(1 - s_t)^{1-\rho} + p_t V_t s_t^{1-\rho}]. \quad (34)$$

The budget constraint

$$c_t + p_t i_t = w_t n_t + k_t R_t \quad (35)$$

is implied by Walras' law.

Let  $z_t = A_t^{\frac{1}{1-\alpha}} V_t^{\frac{\alpha}{1-\alpha}}$ . Consider the auxiliary variables  $\tilde{c}_t = \frac{c_t}{z_{t-1}}$ ,  $\tilde{k}_t = \frac{k_t}{z_{t-1} V_{t-1}}$ ,  $\tilde{p}_t = p_t V_{t-1}$ ,  $\tilde{i}_t = \frac{i_t}{z_{t-1} V_{t-1}}$ ,  $\tilde{\lambda}_t = \lambda_t z_{t-1} V_{t-1}$ ,  $\tilde{w}_t = \frac{w_t}{z_{t-1}}$ ,  $\tilde{R}_t = R_t V_{t-1}$ . Substituting these expressions into equations (26)–(34), one obtains the following equations, which are stationary in the auxiliary variables:<sup>19</sup>

$$\tilde{\lambda}_t = \beta E_t \left\{ \frac{z_{t-1} V_{t-1}}{z_t V_t} \left( \frac{\tilde{R}_{t+1}}{\tilde{c}_{t+1}} + \tilde{\lambda}_{t+1} \left( 1 - \delta + \psi\left(\frac{\tilde{i}_{t+1}}{\tilde{k}_{t+1}}\right) - \psi'\left(\frac{\tilde{i}_{t+1}}{\tilde{k}_{t+1}}\right) \frac{\tilde{i}_{t+1}}{\tilde{k}_{t+1}} \right) \right) \right\} \quad (36)$$

$$\frac{1}{\tilde{c}_t} \tilde{p}_t = \tilde{\lambda}_t \psi'\left(\frac{\tilde{i}_t}{\tilde{k}_t}\right) \quad (37)$$

$$\frac{\tilde{w}_t}{\tilde{c}_t} = \xi n_t^{1/\nu} \quad (38)$$

$$\tilde{k}_{t+1} \frac{z_t V_t}{z_{t-1} V_{t-1}} = (1 - \delta) \tilde{k}_t + \psi\left(\frac{\tilde{i}_t}{\tilde{k}_t}\right) \tilde{k}_t \quad (39)$$

$$\tilde{p}_t \frac{V_t}{V_{t-1}} = \frac{(1 - s_t)^{-\rho}}{s_t^{-\rho}} \quad (40)$$

---

<sup>19</sup>Since the auxiliary variables are independent of  $\rho$ , it follows that the trends in the original variables are not affected by  $\rho$ .

$$\tilde{i}_t \frac{(z_{t-1}V_{t-1})^{1-\alpha}}{A_t V_t} = \tilde{k}_t^\alpha n_t^{1-\alpha} s_t^{1-\rho} \quad (41)$$

$$\tilde{c}_t \frac{z_{t-1}^{1-\alpha} V_{t-1}^{-\alpha}}{A_t} = \tilde{k}_t^\alpha n_t^{1-\alpha} (1-s_t)^{1-\rho} \quad (42)$$

$$\tilde{w}_t \frac{z_{t-1}^{1-\alpha} V_{t-1}^{-\alpha}}{A_t} = (1-\alpha) \tilde{k}_t^\alpha n_t^{-\alpha} \left[ (1-s_t)^{1-\rho} + \tilde{p}_t \frac{V_t}{V_{t-1}} s_t^{1-\rho} \right] \quad (43)$$

$$\tilde{R}_t \frac{(z_{t-1}V_{t-1})^{1-\alpha}}{A_t V_{t-1}} = \alpha \tilde{k}_t^{\alpha-1} n_t^{1-\alpha} \left[ (1-s_t)^{1-\rho} + \tilde{p}_t \frac{V_t}{V_{t-1}} s_t^{1-\rho} \right]. \quad (44)$$

These equations imply that the stationary budget constraint is

$$\tilde{c}_t + \tilde{p}_t \tilde{i}_t = \tilde{w}_t n_t + \tilde{k}_t \tilde{R}_t. \quad (45)$$

When the productivity processes are trend stochastic ( $\rho_a$  and  $\rho_v$  equal to one), the productivity processes (6) and (7) reduce to

$$A_t = \gamma_a A_{t-1} e^{\varepsilon_{a,t}} \quad (46)$$

and

$$V_t = \gamma_v V_{t-1} e^{\varepsilon_{v,t}}. \quad (47)$$

This is because the growth factors  $\frac{A_t}{A_{t-1}}$  and  $\frac{V_t}{V_{t-1}}$  are stationary and thus the trend factors  $\gamma_{1,a}$  and  $\gamma_{1,v}$  are equal to one. Then, equations (36)–(44) further simplify to

$$\tilde{\lambda}_t = \beta E_t \left\{ (\gamma_a \gamma_v e^{\varepsilon_{a,t} + \varepsilon_{v,t}})^{\frac{1}{\alpha-1}} \left( \frac{\tilde{R}_{t+1}}{\tilde{c}_{t+1}} + \tilde{\lambda}_{t+1} \left( 1 - \delta + \psi \left( \frac{\tilde{i}_{t+1}}{\tilde{k}_{t+1}} \right) - \psi' \left( \frac{\tilde{i}_{t+1}}{\tilde{k}_{t+1}} \right) \frac{\tilde{i}_{t+1}}{\tilde{k}_{t+1}} \right) \right) \right\} \quad (48)$$

$$\frac{1}{\tilde{c}_t} \tilde{p}_t = \tilde{\lambda}_t \psi' \left( \frac{\tilde{i}_t}{\tilde{k}_t} \right) \quad (49)$$

$$\frac{\tilde{w}_t}{\tilde{c}_t} = \xi n_t^{1/\nu} \quad (50)$$

$$\tilde{k}_{t+1} (\gamma_a \gamma_v e^{\varepsilon_{a,t} + \varepsilon_{v,t}})^{\frac{1}{1-\alpha}} = (1-\delta) \tilde{k}_t + \psi \left( \frac{\tilde{i}_t}{\tilde{k}_t} \right) \tilde{k}_t \quad (51)$$

$$\tilde{p}_t \gamma_v e^{\varepsilon_{v,t}} = \frac{(1-s_t)^{-\rho}}{s_t^{-\rho}} \quad (52)$$

$$\tilde{i}_t = (\gamma_a \gamma_v e^{\varepsilon_{a,t} + \varepsilon_{v,t}}) \tilde{k}_t^\alpha n_t^{1-\alpha} s_t^{1-\rho} \quad (53)$$

$$\tilde{c}_t = (\gamma_a e^{\varepsilon_{a,t}}) \tilde{k}_t^\alpha n_t^{1-\alpha} (1 - s_t)^{1-\rho} \quad (54)$$

$$\tilde{w}_t = (\gamma_a e^{\varepsilon_{a,t}}) (1 - \alpha) \tilde{k}_t^\alpha n_t^{-\alpha} [(1 - s_t)^{1-\rho} + \tilde{p}_t (\gamma_v e^{\varepsilon_{v,t}}) s_t^{1-\rho}] \quad (55)$$

$$\tilde{R}_t = (\gamma_a e^{\varepsilon_{a,t}}) \alpha \tilde{k}_t^{\alpha-1} n_t^{1-\alpha} [(1 - s_t)^{1-\rho} + \tilde{p}_t (\gamma_v e^{\varepsilon_{v,t}}) s_t^{1-\rho}]. \quad (56)$$

### A.3 Balanced Growth Path with a trend-stationary N-shock and a trend-stochastic I-shock

Identifying the shocks through this framework with curvature, the neutral shock appears to be trend-stationary ( $\rho_a < 1$ ), while the investment one has a stochastic trend.

To detrend the equilibrium condition in this case where the N-shock is stationary and the I-shock is not, let  $z_t = \tilde{A}_t^{\frac{1}{1-\alpha}} V_t^{\frac{\alpha}{1-\alpha}}$ , where

$$\tilde{A}_t = \gamma_a^{\frac{t+1}{1-\rho_a}}. \quad (57)$$

Putting  $\tilde{\gamma}_a = \gamma_a^{\frac{1}{1-\rho_a}}$  and substituting (57) into equations (36)–(44) and putting  $\tilde{a}_t = \frac{A_t}{A_{t-1}}$  one gets the equations

$$\tilde{\lambda}_t = \beta E_t \left\{ (\tilde{\gamma}_a \gamma_v e^{\varepsilon_{v,t}})^{\frac{1}{\alpha-1}} \left( \frac{\tilde{R}_{t+1}}{\tilde{c}_{t+1}} + \tilde{\lambda}_{t+1} \left( 1 - \delta + \psi \left( \frac{\tilde{i}_{t+1}}{\tilde{k}_{t+1}} \right) - \psi' \left( \frac{\tilde{i}_{t+1}}{\tilde{k}_{t+1}} \right) \frac{\tilde{i}_{t+1}}{\tilde{k}_{t+1}} \right) \right) \right\} \quad (58)$$

$$\frac{1}{\tilde{c}_t} \tilde{p}_t = \tilde{\lambda}_t \psi' \left( \frac{\tilde{i}_t}{\tilde{k}_t} \right) \quad (59)$$

$$\frac{\tilde{w}_t}{\tilde{c}_t} = \xi n_t^{1/\nu} \quad (60)$$

$$\tilde{k}_{t+1} (\tilde{\gamma}_a \gamma_v e^{\varepsilon_{v,t}})^{\frac{1}{1-\alpha}} = (1 - \delta) \tilde{k}_t + \psi \left( \frac{\tilde{i}_t}{\tilde{k}_t} \right) \tilde{k}_t \quad (61)$$

$$\tilde{p}_t \gamma_v e^{\varepsilon_{v,t}} = \frac{(1 - s_t)^{-\rho}}{s_t^{-\rho}} \quad (62)$$

$$\tilde{i}_t = (\tilde{a}_t \gamma_v e^{\varepsilon_{v,t}}) \tilde{k}_t^\alpha n_t^{1-\alpha} s_t^{1-\rho} \quad (63)$$

$$\tilde{c}_t = \tilde{a}_t \tilde{k}_t^\alpha n_t^{1-\alpha} (1 - s_t)^{1-\rho} \quad (64)$$

$$\tilde{w}_t = \tilde{a}_t (1 - \alpha) \tilde{k}_t^\alpha n_t^{-\alpha} [(1 - s_t)^{1-\rho} + \tilde{p}_t (\gamma_v e^{\varepsilon_{v,t}}) s_t^{1-\rho}] \quad (65)$$

$$\tilde{R}_t = \tilde{a}_t \alpha \tilde{k}_t^{\alpha-1} n_t^{1-\alpha} [(1 - s_t)^{1-\rho} + \tilde{p}_t (\gamma_v e^{\varepsilon_{v,t}}) s_t^{1-\rho}]. \quad (66)$$

From the definition of  $\tilde{a}_t$  and from (7)–(57), it follows that the stochastic process for  $\tilde{a}_t$  is

$$\ln(\tilde{a}_t) = \ln(\gamma_0) - \frac{\rho_a}{1 - \rho_a} \ln(\gamma_a) + \rho_a \ln(\tilde{a}_{t-1}) + \varepsilon_{a,t}.^{20} \quad (67)$$

---

<sup>20</sup>  $\frac{A_t}{\tilde{A}_{t-1}} = \frac{\gamma_0 A_{t-1}^{\rho_a} \gamma_a^t e^{\varepsilon_{a,t}}}{\gamma_a^{\frac{t}{1-\rho_a}}} = \gamma_0 \left( \frac{A_{t-1}}{\tilde{A}_{t-2}} \right)^{\rho_a} \gamma_a^{\frac{-\rho_a}{1-\rho_a}} e^{\varepsilon_{a,t}}.$